Question 6

Let f be the function defined by $f(x) = k\sqrt{x} - \ln x$ for $x > 0,$ where k is a positive constant.

(a) Find $f'(x)$ and $f''(x)$.

(b) For what value of the constant k does f have a critical point at $x = 1$? For this value of $k,$ determine whether f has a relative minimum, relative maximum, or neither at $x = 1.$ Justify your answer.

(c) For a certain value of the constant $k,$ the graph of f has a point of inflection on the x-axis. Find this value of $k.$

(a) $f'(x) = \dfrac{k}{2\sqrt{x}} - \dfrac{1}{x}$

$f''(x) = -\dfrac{1}{4}kx^{-3/2} + x^{-2}$

$2 : \begin{cases} 1 : f'(x) \\ 1 : f''(x) \end{cases}$

(b) $f'(1) = \dfrac{1}{2}k - 1 = 0 \Rightarrow k = 2$

When $k = 2,$ $f'(1) = 0$ and $f''(1) = -\dfrac{1}{2} + 1 > 0.$

f has a relative minimum value at $x = 1$ by the Second Derivative Test.

$4 : \begin{cases} 1 : \text{sets } f'(1) = 0 \text{ or } f'(x) = 0 \\ 1 : \text{solves for } k \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(c) At this inflection point, $f''(x) = 0$ and $f(x) = 0.$

$f''(x) = 0 \Rightarrow \dfrac{-k}{4x^{3/2}} + \dfrac{1}{x^2} = 0 \Rightarrow k = \dfrac{4}{\sqrt{x}}$

$f(x) = 0 \Rightarrow k\sqrt{x} - \ln x = 0 \Rightarrow k = \dfrac{\ln x}{\sqrt{x}}$

Therefore, $\dfrac{4}{\sqrt{x}} = \dfrac{\ln x}{\sqrt{x}}$

$\Rightarrow 4 = \ln x$

$\Rightarrow x = e^4$

$\Rightarrow k = \dfrac{4}{e^2}$

$3 : \begin{cases} 1 : f''(x) = 0 \text{ or } f(x) = 0 \\ 1 : \text{equation in one variable} \\ 1 : \text{answer} \end{cases}$