AP® CALCULUS AB
2007 SCORING GUIDELINES (Form B)

Question 6

Let f be a twice-differentiable function such that $f(2) = 5$ and $f(5) = 2$. Let g be the function given by $g(x) = f(f(x))$.

(a) Explain why there must be a value c for $2 < c < 5$ such that $f'(c) = -1$.

(b) Show that $g'(2) = g'(5)$. Use this result to explain why there must be a value k for $2 < k < 5$ such that $g''(k) = 0$.

(c) Show that if $f''(x) = 0$ for all x, then the graph of g does not have a point of inflection.

(d) Let $h(x) = f(x) - x$. Explain why there must be a value r for $2 < r < 5$ such that $h(r) = 0$.

(a) The Mean Value Theorem guarantees that there is a value c, with $2 < c < 5$, so that

$$f'(c) = \frac{f(5) - f(2)}{5 - 2} = \frac{2 - 5}{5 - 2} = -1.$$

$$2 : \begin{cases} 1 : \dfrac{f(5) - f(2)}{5 - 2} \\ 1 : \text{conclusion, using MVT} \end{cases}$$

(b) $g'(x) = f'(f(x)) \cdot f'(x)$

$g'(2) = f'(f(2)) \cdot f'(2) = f'(5) \cdot f'(2)$

$g'(5) = f'(f(5)) \cdot f'(5) = f'(2) \cdot f'(5)$

Thus, $g'(2) = g'(5)$.

Since f is twice-differentiable, g' is differentiable everywhere, so the Mean Value Theorem applied to g' on $[2, 5]$ guarantees there is a value k, with $2 < k < 5$, such that $g''(k) = \dfrac{g'(5) - g'(2)}{5 - 2} = 0$.

$$3 : \begin{cases} 1 : g'(x) \\ 1 : g'(2) = f'(5) \cdot f'(2) = g'(5) \\ 1 : \text{uses MVT with } g' \end{cases}$$

(c) $g''(x) = f''(f(x)) \cdot f'(x) \cdot f'(x) + f'(f(x)) \cdot f''(x)$

If $f''(x) = 0$ for all x, then

$g''(x) = 0 \cdot f'(x) \cdot f'(x) + f'(f(x)) \cdot 0 = 0$ for all x.

Thus, there is no x-value at which $g''(x)$ changes sign, so the graph of g has no inflection points.

OR

If $f''(x) = 0$ for all x, then f is linear, so $g = f \circ f$ is linear and the graph of g has no inflection points.

$$2 : \begin{cases} 1 : \text{considers } g'' \\ 1 : g''(x) = 0 \text{ for all } x \end{cases}$$

OR

$$2 : \begin{cases} 1 : f \text{ is linear} \\ 1 : g \text{ is linear} \end{cases}$$

(d) Let $h(x) = f(x) - x$.

$h(2) = f(2) - 2 = 5 - 2 = 3$

$h(5) = f(5) - 5 = 2 - 5 = -3$

Since $h(2) > 0 > h(5)$, the Intermediate Value Theorem guarantees that there is a value r, with $2 < r < 5$, such that $h(r) = 0$.

$$2 : \begin{cases} 1 : h(2) \text{ and } h(5) \\ 1 : \text{conclusion, using IVT} \end{cases}$$