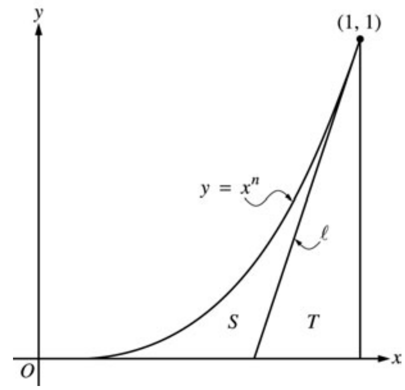
Question 6



Let ℓ be the line tangent to the graph of $y = x^n$ at the point $(1, 1)$, where $n > 1$, as shown above.

(a) Find $\int_0^1 x^n\, dx$ in terms of n.

(b) Let T be the triangular region bounded by ℓ, the x-axis, and the line $x = 1$. Show that the area of T is $\frac{1}{2n}$.

(c) Let S be the region bounded by the graph of $y = x^n$, the line ℓ, and the x-axis. Express the area of S in terms of n and determine the value of n that maximizes the area of S.

(a) $\int_0^1 x^n\, dx = \left.\frac{x^{n+1}}{n+1}\right|_0^1 = \frac{1}{n+1}$

$2: \begin{cases} 1 : \text{antiderivative of } x^n \\ 1 : \text{answer} \end{cases}$

(b) Let b be the length of the base of triangle T.

$\frac{1}{b}$ is the slope of line ℓ, which is n

$\text{Area}(T) = \frac{1}{2}b(1) = \frac{1}{2n}$

$3: \begin{cases} 1 : \text{slope of line } \ell \text{ is } n \\ 1 : \text{base of } T \text{ is } \frac{1}{n} \\ 1 : \text{shows area is } \frac{1}{2n} \end{cases}$

(c) $\text{Area}(S) = \int_0^1 x^n\, dx - \text{Area}(T)$

$= \frac{1}{n+1} - \frac{1}{2n}$

$\frac{d}{dn}\text{Area}(S) = -\frac{1}{(n+1)^2} + \frac{1}{2n^2} = 0$

$2n^2 = (n+1)^2$

$\sqrt{2}\, n = (n+1)$

$n = \frac{1}{\sqrt{2}-1} = 1 + \sqrt{2}$

$4: \begin{cases} 1 : \text{area of } S \text{ in terms of } n \\ 1 : \text{derivative} \\ 1 : \text{sets derivative equal to } 0 \\ 1 : \text{solves for } n \end{cases}$